Perturbation expansion in phase-ordering kinetics. II. *n*-vector model

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The perturbation theory expansion presented earlier to describe the phase-ordering kinetics in the case of a nonconserved scalar order parameter is generalized to the case of the *n*-vector model. At lowest order in this expansion, as in the scalar case, one obtains the theory due to Ohta, Jasnow, and Kawasaki (OJK). The second-order corrections for the nonequilibrium exponents are worked out explicitly in *d* dimensions and as a function of the number of components *n* of the order parameter. In the formulation developed here the corrections to the OJK results are found to go to zero in the large *n* and *d* limits. Indeed, the large-*d* convergence is exponential.

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I. INTRODUCTION

A perturbation theory expansion for treating the scaling features of phase-ordering kinetics in unstable systems [1] was presented earlier [2] for the case of a nonconserved scalar order parameter. In this paper this method is extended to the case of a vector order parameter with *n* components. It is found that spin-wave degrees of freedom introduce some new elements into the theory. However, as in the scalar case, the Ohta, Jasnow, and Kawasaki (OJK) theory [3,1] emerges as the zeroth-order approximation and at this order the nonequilibrium indices λ and ν do not depend on the number of components *n* of the order parameter. The second-order corrections to these exponents are determined as functions of *d* and *n*, and these corrections vanish for both large *d* and *n*.

The theory developed in paper I and here is a two-step process which builds on earlier work [4]. First one maps the original problem for the order parameter $\vec{\psi}$ onto one for an auxiliary field $\vec{m}$. The field configurations associated with the properly chosen auxiliary field are smoother and therefore easier to treat than for the order parameter. The second step is to treat the nonlinear field theory satisfied by $\vec{m}$. It is found in both scalar and continuous $(n>1)$ cases that one must construct, as one constructs a fixed-point Hamiltonian in critical phenomena, the equation of motion satisfied by the auxiliary field in the scaling regime. A unique aspect in the development is the conjecture that there is a general relation $\nu=2\lambda-d$ connecting the nonequilibrium indices. This relation, and the self-consistent maintenance of the $t^{1/2}$ growth law for this problem, fixes the form of the auxiliary field equation of motion at second order in the perturbation theory. An important consequence of this procedure is that the second-order contributions to the indices are exponentially small in the large-*d* limit. In the continuous case it is only the longitudinal part of the auxiliary field equation of motion which must be determined self-consistently. The transverse part is unambiguously determined by consideration of the spin-wave degrees of freedom.

The nature of the perturbation expansion introduced in paper I is elucidated further here. For general *n* it can be seen that one can develop an expansion of the nonlinear terms in the auxiliary field equations of motion in terms of a set of vertices labeled by its number of spin labels [5]. A vertex with ℓ labels can be self-consistently taken to be of $O(\ell/2-1)$. This, in turn, leads to the result that the ℓth-order cumulant is also of order $Oc(\ell/2-1)$. We will refer to this expansion as the *vertex expansion*. As in the scalar case, this expansion is well behaved in the lowest orders of perturbation theory. We obtain nontrivial values for the nonequilibrium exponents at second order in perturbation theory by exponentiating diverging logarithms that are driven, for general dimensionality *d*, by internal time integrations.

II. OVERVIEW

We study here the phase-ordering kinetics generated by the time-dependent Ginzburg-Landau (TDGL) model satisfied by a nonconserved vector order parameter $\psi_i(\vec{r},t)$ $(i=1,2,\ldots,n)$,

$$\frac{\partial\psi_i}{\partial t}=-\Gamma\frac{\delta F}{\delta\psi_i}+\eta_i, \tag{1}$$

where Γ is a kinetic coefficient, *F* is a Ginzburg-Landau effective free energy assumed to be of the form

$$F=\int d^dr\left(\frac{c}{2}\sum_{i=1}^{n}\sum_{\alpha=1}^{d}(\nabla_\alpha\psi_i)^2+V(\vec{\psi})\right), \tag{2}$$

where $c>0$ and the potential *V* is assumed to be of the symmetric degenerate wine-bottle form, $V=V(\vec{\psi}^2)$. We expect only these general properties of *V* will be important in our analysis. $\vec{\eta}$ is a thermal noise that is related to Γ by a fluctuation-dissipation theorem. We assume that the quench is from a high temperature $(T_I>T_c)$, where the system is disordered, to zero temperature, where the noise can be set to zero $(\vec{\eta}=\vec{0})$. It is believed [1] that our final results are independent of the exact nature of the initial state, provided it is a disordered state with short-ranged correlations.

If we rescale length and times we can put our equation of motion in the dimensionless form

$$\Lambda(1)\psi_i(1)=-V_i(\vec{\psi}(1))\equiv-\frac{\partial V(\vec{\psi}(1))}{\partial\psi_i(1)}, \tag{3}$$

where the diffusion operator

$$\Lambda(1)=\frac{\partial}{\partial t_1}-\nabla_1^2 \tag{4}$$

is introduced along with the short-hand notation that 1 denotes $(\mathbf{r}_1,t_1)$. The standard ψ^4 form for the potential is given by $V=-\frac{1}{2}\vec{\psi}^2+\frac{1}{4}(\vec{\psi}^2)^2$.

For late times, following a quench from the disordered to the ordered phase, the order-parameter dynamics obey scaling [1] governed by a single growing length, $L(t)$, which is characteristic of the spacing between defects. In this scaling regime the order-parameter correlation function has a universal scaling form

$$C(12)\equiv\langle\vec{\psi}(1)\cdot\vec{\psi}(2)\rangle=\psi_0^2\mathcal{F}(x,t_1/t_2),$$

where ψ_0 is the magnitude of the order parameter in the ordered phase. The scaled length x is defined as $\vec{x}=(\vec{r}_1-\vec{r}_2)/L(T)$ where, for the nonconserved order-parameter case considered here, the growth law [6] goes as $L(T)\sim T^{1/2}$ where $T=\frac{1}{2}(t_1+t_2)$. In the case of the autocorrelation function $\vec{r}_1=\vec{r}_2=\vec{r}$ we have [7]

$$\langle\vec{\psi}(\vec{r},t_1)\cdot\vec{\psi}(\vec{r},t_2)\rangle\approx\left(\frac{\sqrt{t_1t_2}}{T}\right)^{\lambda}, \tag{5}$$

where λ is a nontrivial nonequilibrium exponent that enters when either t_1 or t_2 is much larger than the other. At equal times the scaling function $F(x)\equiv\mathcal{F}(x,1)$ is a nonanalytic function of x for small $|x|$. This is best reflected in the generalized form of Porod's law [8,9] as expressed in terms of the Fourier transform $\mathcal{F}(Q)\approx Q^{-(n+d)}$ for large scaled wave number Q. The large-x behavior of the scaling function can, with proper definition of x, be put in the form $\mathcal{F}(x)\approx(1/x^{\nu})e^{-(1/2)x^2}$ where ν is a nontrivial subdominant exponent introduced in Ref. [4].

Values for λ and ν were found in paper I which suggested that they are independent. In earlier work in Refs. [10,11], within the approximation developed in Ref. [4], the two exponents were related by

$$\nu=2\lambda-d. \tag{6}$$

We conjecture here that the relation given by Eq. (6) holds more generally and the perturbation theory must be constructed to respect this result. The argument leading to this conjecture follows from the assumption that in the bulk scaling regime and away from defects the only available vector is $\vec{\psi}(1)$ and the potential contribution to the order-parameter equation of motion must be of the form

$$V_i(\vec{\psi}(1))=-\frac{a}{L^2(1)}\psi_i(1), \tag{7}$$

where a is an unknown constant and, if we are to have scaling, the coefficient in Eq. (7) must be proportional to L^{-2}. In Refs. [4] and [10] it was found that $a=\pi/2$. With these assumptions one can follow the development in Ref. [4] to show that the exponent ν can be written in the form

$$\nu=d-\frac{a}{\mu}, \tag{8}$$

where μ is the eigenvalue that plays a prominent part in the theory developed in Ref. [4]. Similarly, generalizing the analysis in Ref. [10] for two-time correlation functions in the limit of large time separations, one can show that the nonequilibrium exponent λ is given by

$$\lambda=d-\frac{a}{2\mu}. \tag{9}$$

By eliminating $a/2\mu$ between Eqs. (8) and (9) we obtain the scaling relation given by Eq. (6).

III. AUXILIARY FIELD METHOD FOR THE *n*-VECTOR MODEL

Our goal here is to generalize the perturbation theory expansion approach developed in paper I for a scalar order parameter to the case of general n. We will again express the order parameter as a sum of two pieces

$$\vec{\psi}=\vec{\sigma}+\vec{u}, \tag{10}$$

where, as in Ref. [12], $\vec{\sigma}$ is the ordering component of the order parameter that depends locally on an n-component field $\vec{m}$: $\vec{\sigma}=\vec{\sigma}(\vec{m})$. The field $\vec{u}$ represents the fluctuations about the ordered configuration. The functional dependence of $\vec{\sigma}$ on $\vec{m}$ is determined as a solution of the Euler-Lagrange equation for the associated stationary defect problem

$$\sum_{\ell}\frac{\partial^2\sigma_i}{\partial m_\ell^2}=V_i(\sigma(m)), \tag{11}$$

where $\vec{m}$ is taken to be the coordinate and the solution must satisfy the boundary condition $\lim_{|m|\to\infty}\sigma^2=\psi_0^2$. If we introduce the notation

$$\sigma_{i;j_1j_2\ldots j_\ell}\equiv\frac{\partial^\ell\sigma_i}{\partial m_{j_1}\partial m_{j_2}\cdots\partial m_{j_\ell}}$$

then Eq. (11) reads

$$\sum_{\ell}\sigma_{i;\ell\ell}=V_i(\sigma(m)). \tag{12}$$

One can obtain the defect profile analytically for the case of a general degenerate potential for the scalar order-parameter case. In the particular case of a ψ^4 potential, one obtains the usual interfacial kink solution $\sigma[m]=\tanh(m/\sqrt{2})$. For systems with continuous symmetry, $n>1$, one does not have a closed form solution for the defect profile even for the ψ^4 potential. However, one can make some general statements about the form of the profile in the ordered bulk regime. For the lowest *charge* defects we can write the order-parameter profile in the form $\vec{\sigma}[\vec{m}]=A(m)\hat{m}$, then the Euler-Lagrange equation reduces to an equation for the amplitude $A(m)$ given by

$$A''+\frac{(n-1)}{m}\left(A'-\frac{A}{m}\right)=\frac{\partial V(A)}{\partial A}. \qquad (13)$$

Solutions of Eq. (13) for large m, in the bulk away from defect cores, in contrast to the scalar case which has an exponential approach to the ordered value, show an algebraic approach to the ordered value

$$A=\psi_0\left(1-\frac{\xi^2}{m^2}+\cdots\right), \qquad (14)$$

where $\xi^2\equiv(n-1)/V''(\psi_0)$. This solution requires that the ordered value of the magnitude of the order parameter be given by the solution to $V'(\psi_0)=0$, and, for the solution to be stable, we require $V''(\psi_0)=q_0^2>0$. There are some additional general properties we need in the large m regime. If we write

$$V_i(\vec{\sigma})=\frac{\partial V(A)}{\partial A}\frac{\partial A}{\partial \sigma_i}=V'(A)\hat{m}_i, \qquad (15)$$

and use the expansion

$$V'(A)=V'(\psi_0)-\psi_0\frac{\xi^2}{m^2}V''(\psi_0)+\cdots=-\psi_0\frac{(n-1)}{m^2}+\cdots,$$

then we have

$$V_i(\vec{\sigma})=-\hat{m}_i\psi_0\frac{(n-1)}{m^2}+\cdots. \qquad (16)$$

We also need the second derivative of the potential with respect to the order parameter. Taking the derivative of Eq. (15) with respect to σ_j, we obtain

$$V_{ij}(\vec{m})=\delta_{ij}\frac{V'(A)}{A}+\sigma_i\sigma_j\frac{1}{A}\left(\frac{V'(A)}{A}\right)'$$
$$=P_{ij}(\vec{m})\frac{V'(A)}{A}+\hat{m}_i\hat{m}_jV''(A),$$

where the transverse projection operator is defined by

$$P_{ij}(\vec{m})=\delta_{ij}-\hat{m}_i\hat{m}_j.$$

Evaluating $V_{ij}(\vec{m})$ in the bulk, where we can use Eq. (14), we obtain

$$V_{ij}(\vec{m})=q_0^2\hat{m}_i\hat{m}_j-P_{ij}\frac{(n-1)}{m^2}, \qquad (17)$$

where $q_0^2=(n-1)/\xi^2$.

Armed with these results, we next discuss the equations of motion governing the fields $\vec{m}$ and $\vec{u}$. The idea is to separate the original order-parameter equation of motion into equations for $\vec{m}$ and $\vec{u}$ which ensure that we do obtain ordering, fluctuations are small, and the zeros of $\vec{m}$ reflect the zeros of the order parameter $\vec{\psi}$. The condition that $\vec{\sigma}$ govern the ordering requires that in the bulk, away from any defect cores, $\vec{\sigma}^2\rightarrow\psi_0^2$ and $\vec{u}^2\rightarrow 0$. We expect, for both the scalar and continuous cases, that $\vec{u}$ controls the transition region between the defect core and the ordered bulk.

Let us look first at the simpler scaler order-parameter case. If the form given by Eq. (10) is inserted into the equation of motion given by Eq. (4), we obtain, without approximation, the equation of motion for u,

$$\Lambda(1)u(1)+\sigma_1(1)\Lambda(1)m(1)=-V'[\sigma(1)+u(1)]$$
$$+\sigma_2(1)[\nabla m(1)]^2. \qquad (18)$$

We assume that the equation of motion satisfied by m is of the form,

$$\Lambda(1)m(1)=\Xi(1), \qquad (19)$$

where $\Xi(1)$ is a functional of $m(1)$ and, if naive scaling is to hold, we require $\Xi\approx\mathcal{O}(L^{-1})$. How one proceeds when one does not have naive scaling will be discussed elsewhere. Using Eq. (19) in Eq. (18) for u leads to an equation of motion for the field $u(1)$,

$$\Lambda(1)u(1)=-V'[\sigma(1)+u(1)]+\sigma_2(1)[\nabla m(1)]^2$$
$$-\sigma_1(1)\Xi(1). \qquad (20)$$

Our goal then is to show that we can choose $\Xi(1)$ such that u is small in the bulk and near a defect. More quantitatively, in the bulk, we have $\sigma[m]\rightarrow\psi_0\,\mathrm{sgn}[m]$, while $\lim_{|m|\rightarrow\infty}u[m]=0$. To see how this works and to put some constraints on $\Xi(1)$, we note, if u is small in the bulk, then we can self-consistently expand

$$V'[\sigma(1)+u(1)]=V'[\sigma(1)]+V''[\sigma(1)]u(1)+\cdots$$
$$=\sigma_2(1)+q_0^2u(1)+\cdots.$$

The equation of motion for u then takes the form in the bulk

$$[\Lambda(1)+q_0^2]u(1)=-\sigma_2(1)\{1-[\nabla m(1)]^2\}-\sigma_1(1)\Xi(1). \qquad (21)$$

In working in the bulk ordered regime, which makes the dominate contribution to the scaling properties, we can estimate $m\approx L$, $\nabla\approx L^{-1}$, and $\partial/\partial t\approx L^{-2}$. Notice on the left-hand side of Eq. (21) that u has acquired a *mass* and, in the long-time long-distance limit, the term where u is multiplied by a constant dominates the derivative terms and u is given by

$$q_0^2u(1)=-\sigma_2(1)\{1-[\nabla m(1)]^2\}-\sigma_1(1)\Xi(1).$$

In the limit of large $|m|$ the derivatives of σ go exponentially to zero and the right-hand side of Eq. (21) is exponentially small. Clearly we can construct a solution for u which is also exponentially small in the bulk. We have then on rather general principles that the field u must vanish rapidly as one moves into the bulk away from interfaces. We then have the following rather weak constraints on Ξ.

(i) Ξ must be chosen such that $\vec{m}$ grows and the field $\vec{\sigma}$ orders.

(ii) If the system satisfies naive scaling then Ξ must go as $O(L^{-1})$ in the bulk.

The form for Ξ in the bulk which fulfills these requirements is given by

$$\Xi(1)=\mathrm{sgn}(m(1))\{g_0(1)+g_1(1)[\nabla m(1)]^2+\cdots\},$$

where $g_0(1)$ and $g_1(1)$ must go as L^{-1} for long times.

Let us now look at how this separation process carries over to the case of a continuous order parameter $n>1$. Inserting Eq. (10) into Eq. (3) generates an equation of motion for $\vec{u}(1)$ of the form

$$\begin{aligned}\Lambda(1)u_i(1)&=-V_i(\vec{\sigma}(1)+\vec{u}(1))-\Lambda(1)\sigma_i(1)\\&=-V_i(\vec{\sigma}(1)+\vec{u}(1))-\sigma_{i;j}\Lambda(1)m_j(1)\\&\quad+\sigma_{i;jk}\nabla_\alpha m_j\nabla_\alpha m_k.\end{aligned}\tag{22}$$

We assume that $\vec{m}$ satisfies the nonlinear equation of motion

$$\Lambda(1)\vec{m}(1)=\vec{\Xi}(1),\tag{23}$$

where $\vec{\Xi}$ has the same interpretation as in the scalar case. Inserting Eq. (23) back into Eq. (22) gives the basic equation of motion for $\vec{u}$,

$$\begin{aligned}\Lambda(1)u_i(1)=&-V_i(\vec{\sigma}(1)+\vec{u}(1))-\sigma_{i;j}\Xi_j(1)\\&+\sigma_{i;jk}\nabla_\alpha m_j\nabla_\alpha m_k.\end{aligned}\tag{24}$$

In the bulk we self-consistently assume that the potential can be expanded in powers of $\vec{u}$,

$$V_i(\vec{\sigma}(1)+\vec{u}(1))=V_i(\vec{\sigma}(1))+V_{ij}(\vec{\sigma}(1))u_j(1)+\cdots.\tag{25}$$

The two terms on the right-hand side of Eq. (25) can be simplified using Eqs. (16) and (17) and lead to the new form for Eq. (24):

$$\begin{aligned}\Lambda(1)u_i(1)=&\frac{(n-1)}{m^2}\hat{m}_i-q_0^2\hat{m}_iu_L+\frac{(n-1)}{m^2}u_i^T-\sigma_{ij}\Xi_j(1)\\&+\sigma_{i;jk}\nabla_\alpha m_j\nabla_\alpha m_k,\end{aligned}\tag{26}$$

where we have divided $\vec{u}$ into its longitudinal and transverse parts:

$$u_i=\hat{m}_iu_L+u_i^T,\tag{27}$$

where $\hat{m}\cdot\vec{u}^T=0$. Dotting $\hat{m}_i$ into Eq. (26), we obtain the equation determining the longitudinal part of $\vec{u}$,

$$\begin{aligned}\hat{m}_i\Lambda(1)u_i(1)=&\frac{(n-1)}{m^2}-q_0^2u_L+\hat{m}_i\sigma_{i;jk}\nabla_\alpha m_j\nabla_\alpha m_k\\&-\hat{m}_i\sigma_{i;j}\Xi_j.\end{aligned}\tag{28}$$

In the bulk, starting with $\psi_i=\psi_0\hat{m}_i$, we easily obtain

$$\sigma_{i;j}=\frac{\psi_0}{m}P_{ij}[\vec{m}],$$

$\hat{m}_i\sigma_{i;j}\approx O(L^{-3})$, $\hat{m}_i\sigma_{i;j}\Xi_j\approx O(L^{-4})$, and the term proportional to $\hat{m}_i\sigma_{i,j}$ can be dropped in Eq. (28) compared to the terms of $O(1/m^2)$. Next, since

$$\sigma_{i;jk}=-\frac{\psi_0}{m^2}(\delta_{ij}\hat{m}_k+\delta_{ik}\hat{m}_j+\delta_{jk}\hat{m}_i-3\hat{m}_i\hat{m}_j\hat{m}_k),\tag{29}$$

in the bulk, we have

$$\hat{m}_i\sigma_{i;jk}\nabla_\alpha m_j\nabla_\alpha m_k=-\frac{\psi_0}{m^2}P_{jk}\nabla_\alpha m_j\nabla_\alpha m_k.$$

Using this result back in Eq. (28), the equation of motion in the bulk for the longitudinal part of the fluctuation field is given then by

$$\hat{m}_i\Lambda(1)u_i(1)=\frac{(n-1)}{m^2}-q_0^2u_L-\frac{\psi_0}{m^2}P_{jk}\nabla_\alpha m_j\nabla_\alpha m_k.\tag{30}$$

All derivative terms in Eq. (30) acting on $\vec{u}$ can be dropped in comparison with the $q_0^2u_L$ term in the scaling regime, and, to lowest order in L^{-1}, we can express u_L explicitly in terms of $\vec{m}$:

$$q_0^2u_L=\frac{1}{m^2}[(n-1)-P_{jk}\nabla_\alpha m_j\nabla_\alpha m_k].\tag{31}$$

Because of the mass term, $q_0^2>0$, we see indeed that u_L is of order L^{-2} and no additional constraints are put on $\vec{\Xi}$.

Let us turn next to the transverse part of $\vec{u}$. Multiplying Eq. (29) by the transverse projector P we obtain

$$\begin{aligned}P_{ij}\Lambda(1)u_j(1)=&\frac{(n-1)}{m^2}u_i^T-P_{ik}\sigma_{kj}\Xi_j(1)\\&+P_{i\ell}\sigma_{\ell;jk}\nabla_\alpha m_j\nabla_\alpha m_k.\end{aligned}\tag{32}$$

Then in the bulk

$$P_{ik}\sigma_{kj}\Xi_j(1)=\frac{\psi_0}{m}P_{ij}\Xi_j(1)=\frac{\psi_0}{m}\Xi_i^T(1),\tag{33}$$

and, using Eq. (29),

$$P_{i\ell}\sigma_{\ell;jk}\nabla_\alpha m_j\nabla_\alpha m_k=-2\frac{\psi_0}{m^2}(\hat{m}_k\nabla_\alpha m_k)P_{ij}\nabla_\alpha m_j.\tag{34}$$

Setting Eqs. (33) and (34) back into Eq. (32) gives

$$\begin{aligned}P_{ij}\Lambda(1)u_j(1)=&\frac{(n-1)}{m^2}u_i^T-\frac{\psi_0}{m}\Xi_i^T(1)\\&-2\frac{\psi_0}{m^2}(\hat{m}_k\nabla_\alpha m_k)P_{ij}\nabla_\alpha m_j\end{aligned}$$

which governs the transverse fluctuations. For self-consistency $\vec{u}^T$ must be small in the bulk. This requires us to choose the transverse component of $\vec{\Xi}$ to be given by

$$-\frac{\psi_0}{m}\Xi_j^T(1)-2\frac{\psi_0}{m^2}(\hat{m}_k\nabla_\alpha m_k)P_{ij}\nabla_\alpha m_j=0$$

or

$$\Xi_i^T(1)=-\frac{2}{m}(\hat{m}_k\nabla_\alpha m_k)P_{ij}\nabla_\alpha m_j. \quad (35)$$

With this choice, the equation for the transverse fluctuations is given by

$$P_{ij}\Lambda(1)u_j(1)=\frac{(n-1)}{m^2}u_i^T.$$

Thus we have that u_i^T is generated by any coupling back to $u_L\approx O(1/L^2)$ via $P_{ij}\Lambda(1)u_j(1)$. We can estimate

$$u_i^T\approx L^2P_{ij}\Lambda(1)u_L(1)\approx O(1/L^2),$$

and generally $\vec{u}\approx O(1/L^2)$. The requirement that the bulk part of the transverse fluctuations $\vec{u}^T$ be small fixes the form of $\vec{\Xi}^T$ to be given by Eq. (35). This form does not depend on any details of the potential, and can be simplified. Consider

$$\frac{2}{m}\hat{m}_k\nabla_\alpha m_k=\frac{1}{m^2}\nabla_\alpha m^2=\frac{2}{m}\nabla_\alpha m$$

and

$$P_{ij}\nabla_\alpha m_j=m\frac{\partial\hat{m}_i}{\partial m_j}\nabla_\alpha m_j=m\nabla_\alpha\hat{m}_i.$$

Inserting these last two results back into Eq. (35) gives

$$\Xi_i^T(1)=-\frac{2}{m}(\nabla_\alpha m)m\nabla_\alpha\hat{m}_i=-2\nabla_\alpha m\nabla_\alpha\hat{m}_i. \quad (36)$$

The equation of motion satisfied by $\vec{m}$ is given by Eq. (23). While the transverse part of $\vec{\Xi}$ is given by Eq. (36), the longitudinal part of $\vec{\Xi}$ is constrained only by the requirement that it scale as L^{-1}. The precise form for Ξ_L in the scaling regime must be determined self-consistently within perturbation theory. If we look at the building blocks in the problem we see that the quantities that are of $O(1)$ are $\hat{m}$ and $\nabla_\alpha m_i$. Thus one sees that the structure of the longitudinal part of the equation of motion in the bulk scaling regime can be assumed to be of the general form:

$$\begin{aligned}\Xi_i^L(1)=\hat{m}_i(1)\{&g_0(1)+g_1(1)[\nabla_\alpha m_j(1)]^2\\&+g_{sjk\ell}^{(2)}(1)\nabla_\alpha m_s(1)\nabla_\alpha m_j(1)\nabla_\beta m_k(1)\\&\times\nabla_\beta m_\ell(1)+\cdots\}.\end{aligned}$$

Clearly in the long-time limit we require the g's be of $O(L^{-1})$ and, as we shall see, that we can self-consistently construct the g_p's if we assume that $g_p=O(p)$ in the vertex expansion. Our final results at second order in our expansion will depend on g_0 and g_1.

The assumption we make here is that the higher-order terms proportional to $g^{(\ell)}$, for $\ell>0$, contribute in a nontrivial way starting at order $\ell+2$. Thus $g^{(2)}$, due to various contractions, act at second order only to renormalize g_0.

IV. FIELD THEORY FOR AUXILIARY FIELD

The equation of motion satisfied by the ordering field $\vec{m}$ including terms which contribute up to second order is given in the bulk by

$$\Lambda(1)m_i(1)=g_0(t_1)\hat{m}_i(1)+\tilde{\Xi}_i(1), \quad (37)$$

where $\tilde{\Xi}_i(1)=\tilde{\Xi}_i^L(1)+\Xi_i^T(1)$, and

$$\tilde{\Xi}_i^L(1)=g_1(t_1)\hat{m}_i(1)[\nabla_\alpha m_j(1)]^2,$$

$$\Xi_i^T(1)=-2\nabla_\alpha m(1)\nabla_\alpha\hat{m}_i(1).$$

The functions $g_0(t_1)$ and $g_1(t_1)$ are determined within perturbation theory. Our analysis will follow the standard Martin-Siggia-Rose (MSR) [13] method in its functional integral form as developed by DeDominicis and Peliti [14] and presented in detail in paper I. In the MSR method the field theoretical development requires a doubling of fields to include the response field $\vec{M}$. As in paper I, we introduce a field $\vec{h}(1)$ conjugate to $\vec{m}(1)$ and a field $\vec{H}(1)$ conjugate to $\vec{M}(1)$.

Following closely the formal development in paper I, we find that the fundamental equation satisfied by the average of the field $\vec{m}$, in the presense of sources, is given by

$$i[\Lambda(1)\langle m(1)\rangle_h-Q(1)]=-\int d2\,\Pi_0(12)\langle M(2)\rangle_h+H(1) \quad (38)$$

where the vector labels are suppressed,

$$\Pi_0^{ij}(12)\equiv\delta(t_1-t_0)\delta(t_1-t_2)g(\vec{r}_1-\vec{r}_2)\delta_{ij}.$$

It is assumed here that the field $\vec{m}(1)$, at the initial time t_0 has Gaussian statistics with variance

$$\langle m_0^i(\vec{r}_1)m_0^j(\vec{r}_2)\rangle=\delta_{ij}g(\vec{r}_1-\vec{r}_2).$$

The nonlinear vertices in Eq. (38) are given by

$$Q_i(1)=\langle\Xi_i(1)\rangle\equiv Q_i^D(1)+Q_i^L(1)+Q_i^T(1),$$

with

$$Q_i^D(1)=\langle\Xi_i^D(1)\rangle=g_0(1)\langle\hat{m}_i(1)\rangle,$$

$$Q_i^L(1)=\langle\Xi_i^L(1)\rangle=g_1(1)\langle\hat{m}_i(1)[\nabla m_j(1)]^2\rangle,$$

$$Q_i^T(1)=\langle\Xi_i^T(1)\rangle=(-2)\langle\nabla_\alpha m(1)\nabla_\alpha\hat{m}_i(1)\rangle.$$

The fundamental equation satisfied by the average of the MSR response field is given by

$$-i[\tilde{\Lambda}(1)\langle M(1)\rangle_h+\tilde{Q}(1)]=h(1), \tag{39}$$

where we define

$$\tilde{\Lambda}(1)=\frac{\partial}{\partial t_1}+\nabla_1^2,$$

and the nonlinear contributions are given by

$$\tilde{Q}_i(1)=\langle\overline{\Xi}_i(1)\rangle=\tilde{Q}_i^D(1)+\tilde{Q}_i^L(1)+\tilde{Q}_i^T(1),$$

with

$$\tilde{Q}_i^D(1)=\langle\overline{\Xi}_i^D(1)\rangle=g_0(1)\langle M_j(1)\rho_{ij}(1)\rangle,$$

$$\begin{aligned}\tilde{Q}_i^L(1)&=\langle\overline{\Xi}_i^L(1)\rangle\\&=\langle M_j(1)g_1(1)\rho_{ij}(1)[\nabla_\alpha m_k(1)]^2\rangle\\&\quad-\langle\nabla_\alpha[M_j(1)g_1(1)\hat{m}_j(1)2\nabla_\alpha m_i(1)]\rangle\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}_i^T(1)&=\langle\overline{\Xi}_i^T(1)\rangle\\&=\langle 2\hat{m}_i(1)\nabla_\alpha[M_j(1)\nabla_\alpha\hat{m}_j(1)]\\&\quad+2\rho_{ij}(1)\nabla_\alpha[M_j(1)\nabla_\alpha m(1)]\rangle,\end{aligned}$$

and

$$\rho_{ij}(1)=\frac{1}{m}(\delta_{ij}-\hat{m}_i\hat{m}_j).$$

All correlation functions of interest can be generated as functional derivatives of $\langle m(1)\rangle_h$ or $\langle M(1)_h$ with respect to $h(1)$ and $H(1)$. In the limit in which the source fields vanish, each term in the two fundamental equations vanish. Therefore it is derivatives with respect to the external sources of these equations which are of interest. Let us introduce the notation that $G_{A_1,A_2,\ldots,A_n}(1,2,\ldots n)$ is the nth order cumulant for the set of fields $\{A_1,A_2,\ldots,A_n\}$, where field A_1 has argument (1), field A_2 has argument (2), etc. This notation is needed when have cumlants with m and M mixed. As an example,

$$G_{Mmmm}(1234)=\frac{\delta^3\langle m(4)\rangle_h}{\delta H(1)\,\delta h(2)\,\delta h(3)}. \tag{40}$$

As a short hand for cumulants involving only m fields we write

$$G^{(n)}(12\cdots n)=\frac{\delta^{n-1}}{\delta h(n)\,\delta h(n-1)\cdots\delta h(2)}\langle m(1)\rangle_h. \tag{41}$$

The hierarchy of equations connecting these cumulants is given by taking functional derivatives of the fundamental equations given by Eqs. (38) and (39).

The equations governing the nth order cumulants are given by

$$-i[\tilde{\Lambda}(1)G_{Mm\cdots m}(12\cdots n)+\tilde{Q}_n(12\cdots n)]=0 \tag{42}$$

and

$$\begin{aligned}&i[\Lambda(1)G^{(n)}(12\cdots n)-Q_n(12\cdots n)]\\&\quad=-\int d\bar{1}\,\Pi_0(1\bar{1})G_{Mm\cdots m}(\bar{1}2\cdots n),\end{aligned} \tag{43}$$

where the Q's are defined by

$$\tilde{Q}_n(12\cdots n)=\frac{\delta^{n-1}}{\delta h(n)\,\delta h(n-1)\cdots\delta h(2)}\tilde{Q}(1) \tag{44}$$

and

$$Q_n(12\cdots n)=\frac{\delta^{n-1}}{\delta h(n)\,\delta h(n-1)\cdots\delta h(2)}Q(1). \tag{45}$$

With this notation the equations determining the two-point functions can be written as

$$-i[\tilde{\Lambda}(1)G_{Mm}(12)+\tilde{Q}_2(12)]=\delta(12), \tag{46}$$

$$i[\Lambda(1)G(12)-Q_2(12)]=-\int d\bar{1}\,\Pi_0(1\bar{1})G_{Mm}(\bar{1}2). \tag{47}$$

The point now is to show that there is a consistent perturbation expansion for this theory. To get started we need to express $\hat{Q}_i(1)$ and $Q_i(1)$ in terms of a fundamental set of vertices which can be written in terms of the singlet probability distribution

$$P_h(\vec{x},1)=\langle\delta(\vec{x}-\vec{m}(1))\rangle_h.$$

After a great deal of rearrangement one can show that the nonlinear vertices, the Q's can be put in the form

$$Q_i^D(1)=g_0(1)U_i(1), \tag{48}$$

$$Q_i^L(1)=\sum_j\int d\bar{2}d\bar{3}\,g_1(1)w(1\bar{2}\bar{3})O_{jj}(\bar{2}\bar{3})U_i(1),$$

$$Q_i^T(1)=\sum_{jk}\int d\bar{2}d\bar{3}\,w(1\bar{2}\bar{3})O_{jk}(\bar{2}\bar{3})U_{jk,i}(1),$$

$$\tilde{Q}_i^D(1)=-\sum_j\int d\bar{2}d\bar{3}\,g_0(1)O_{H_j}(1)U_{i,j}(1),$$

$$\begin{aligned}\tilde{Q}_i^{L,1}(1)=-\sum_{jk}\int d\bar{2}d\bar{3}\,g_1(1)w(1\bar{2}\bar{3})O_{kk}(\bar{2}\bar{3})\\\times O_{M_j}(1)U_{i,j}(1),\end{aligned}$$

$$\tilde{Q}_i^{L,2}(1) = -\sum_j \int d\bar{2}d\bar{3}\, 2g_1(1)\tilde{w}(1\overline{23})O_i(\bar{2}) \times O_{M_j}(\bar{3})U_j(\bar{3}),$$

$$\tilde{Q}_i^{T,1}(1) = -\sum_{js\ell} \int d\bar{2}d\bar{3}\, w(1\overline{23})O_{M_j}(1) \times O_{s\ell}(\overline{23})\mathcal{Q}_{s\ell,ij}(1),$$

$$\tilde{Q}_i^{T,2}(1) = -\sum_{js\ell} \int d\bar{2}d\bar{3}\, 2\tilde{w}(1\overline{23})O_{M_j}(\bar{3}) \times O_\ell(\bar{2})U_{i\ell,j}(1), \tag{49}$$

where we have introduced the operators

$$O_j(2) = \frac{\delta}{\delta h_j(2)} + G_j^{(1)}(2),$$

and

$$O_{jk}(23) = \frac{\delta^2}{\delta h_j(2)\,\delta h_k(3)} + G_{jk}^{(2)}(23) + G_j^{(1)}(2)\frac{\delta}{\delta h_k(3)} + G_k^{(1)}(3)\frac{\delta}{\delta h_j(2)} + G_j^{(1)}(2)G_k^{(1)}(3).$$

We have also introduced the three-point vertices

$$w(123) = \sum_{\alpha=1}^{d} \nabla_\alpha^{(1)}\delta(12)\nabla_\alpha^{(1)}\delta(13)$$

and

$$\tilde{w}(123) = \nabla_\alpha{}^{(1)}[\delta(13)\nabla_\alpha{}^1\delta(12)].$$

Each term in these expressions for the Q's can be expressed in terms of the set of nonlinear vertices which are integral moments of $P_h[\vec{x}]$:

$$U_{ijk\ldots,\ell mn\ldots}(1) = \int d^n x\; \hat{x}_i\hat{x}_j\hat{x}_k\cdots\nabla_x^\ell\nabla_x^m\nabla_x^n\cdots P_h[\vec{x},1]. \tag{50}$$

We have also defined

$$\mathcal{Q}_{s\ell,ij}(1) = U_{s\ell,ij}(1) - U_{is,\ell j}(1) - U_{i\ell,sj}(1).$$

V. PERTURBATION THEORY EXPANSION

All of the cumulants involving the field $\vec{m}$ can, in principle, be obtained from Eqs. (38) and (39) by taking functional derivatives. This then requires that we work out the functional derivatives of Q_n and $\tilde{Q}_n$ which are defined by Eqs. (44) and (45). These objects are functional derivatives of Q_1 and $\tilde{Q}_1$ which are proportional to a few of the $U_{ijk\ldots,\ell mn\ldots}(1)$ and functional derivatives of these quantities. From this discussion it should be clear that all of the Q_n and $\hat{Q}_n$ can be written as a product of cumulants multiplying vertices given by

$$U_{ijk\ldots,\ell mn\ldots,stu\ldots}(1;234\ldots) = \frac{\delta}{\delta h_s(2)}\frac{\delta}{\delta h_t(3)}\frac{\delta}{\delta h_u(4)}\cdots U_{ijk\ldots,\ell mn\ldots}(1).$$

The point we want to establish is that if U has p vector labels, then, at lowest order, we can take U to be of $\mathcal{O}(p/2-1)$, plus higher-order terms.

The perturbation theory expansion for the $U_{ijk\ldots,\ell mn\ldots}(1)$ follows from the expansion properties of the singlet-distribution function. The perturbation theory expansion for $P_h(\vec{x},1)$ is straightforward. Using the integral representation for the δ function, we have

$$P_h(\vec{x},1) = \int \frac{d^n k}{(2\pi)^n} e^{-i\vec{k}\cdot\vec{x}}\langle e^{\mathcal{H}(1)}\rangle_h,$$

where $\mathcal{H}(1)\equiv i\vec{k}\cdot\vec{m}(1)$. The average of the exponential is precisely of the form which can be rewritten in terms of cumulants:

$$\Phi(\vec{k},1)\equiv\langle e^{\mathcal{H}(1)}\rangle_h = \exp\left[\sum_{s=1}^{\infty}\frac{1}{s!}G_{\mathcal{H}}^{(s)}(1)\right],$$

where $G_{\mathcal{H}}^{(s)}(1)$ is the sth-order cumulant for the field $\mathcal{H}(1)$. Since $\mathcal{H}(1)$ is proportional to $\vec{m}(1)$ these are, up to factors of $i\vec{k}$ to the sth power, just the cumulants for the m field:

$$G_{\mathcal{H}}^{(1)}(1) = i\sum_{\alpha_1} k_{\alpha_1}G_{\alpha_1}^{(1)}(1),$$

$$G_{\mathcal{H}}^{(2)}(1) = (i)^2\sum_{\alpha_1\alpha_2} k_{\alpha_1}k_{\alpha_2}G_{\alpha_1\alpha_2}^{(2)}(11),$$

$$G_{\mathcal{H}}^{(3)}(1) = (i)^3\sum_{\alpha_1\alpha_2\alpha_3} k_{\alpha_1}k_{\alpha_2}k_{\alpha_3}G_{\alpha_1\alpha_2\alpha_3}^{(3)}(111),$$

and so on. We can therefore write

$$\Phi(\vec{k},1) = \exp\left[\sum_{s=1}^{\infty}\frac{(i)^s}{s!}k_{\alpha_1}k_{\alpha_2}\cdots k_{\alpha_s}G_{\alpha_1\alpha_2\alpha_3\ldots\alpha_s}^{(s)}(11\ldots1)\right].$$

We will assume, as we will show self-consistently, that pth-order cumulants are of order $p/2-1$ in the vertex expansion. Expanding and keeping terms up to the four-point cumulant, we obtain

$$P_h(\vec{x},1) = \left[1 - \sum_{\alpha_1\alpha_2\alpha_3}\frac{1}{3!}G_{\alpha_1\alpha_2\alpha_3}^{(3)}(111)\nabla_x^{\alpha_1}\nabla_x^{\alpha_2}\nabla_x^{\alpha_3} + \cdots\right] \times P_h^{(0)}(\vec{x},1), \tag{51}$$

where

$$P_h^{(0)}(\vec{x},1) = \int \frac{d^n k}{(2\pi)^n}\Phi_h^{(0)}(\vec{k},1)e^{-i\vec{k}\cdot\vec{x}} \tag{52}$$

and

$$\Phi_h^{(0)}(\vec{k},1) = e^{i\Sigma\alpha_1 k_{\alpha_1}G_{\alpha_1}^{(1)}(1)}e^{-1/2\Sigma\alpha_1\alpha_2 k_{\alpha_1}k_{\alpha_2}G_{\alpha_1\alpha_2}^{(2)}(11)}.$$

We can define the lowest order set of vertices

$$U^{(0)}_{ijk\ldots,\ell mn\ldots}(1)=\int d^nx\ \hat{x}_i\hat{x}_j\hat{x}_k\ldots\nabla^{\ell}_x\nabla^m_x\nabla^n_x\ldots P^{(0)}_h[\vec{x},1]$$
$$=\int d^nx\ \hat{x}_i\hat{x}_j\hat{x}_k\ldots\nabla^{\ell}_x\nabla^m_x\nabla^n_x\ldots\int\frac{d^nk}{(2\pi)^n}\Phi^{(0)}_h(\vec{k},1)e^{-i\vec{k}\cdot\vec{x}}$$
$$=\int d^nx\ \hat{x}_i\hat{x}_j\hat{x}_k\ldots\int\frac{d^nk}{(2\pi)^n}(-ik_\ell)(-ik_m)(-ik_n)\ldots\int\frac{d^nk}{(2\pi)^n}\Phi^{(0)}_h(\vec{k},1)e^{-i\vec{k}\cdot\vec{x}}. \tag{53}$$

It should be clear, after inserting Eq. (51) back into Eq. (50), that

$$U_{ijk\ldots,\ell mn\ldots}(1)=U^{(0)}_{ijk\ldots,\ell mn\ldots}(1)+\sum_{\alpha_1\alpha_2\alpha_3\alpha_4}U^{(0)}_{ijk\ldots,\ell mn\ldots\alpha_1\alpha_2\alpha_3\alpha_4}(1)\times\frac{1}{4!}G^{(4)}_{\alpha_1\alpha_2\alpha_3\alpha_4}(1111)+\ldots. \tag{54}$$

It is clear that if the term on the first line of Eq. (54) is of $O(p)$, then the term on the second line is of $O(p+3)$.

For these ideas to be self-consistent then the functional derivatives of $U^{(0)}\ldots$ must be higher order. The reason this works is because factors of the one-point cumulant $G_i(1)$ enters in the exponential appearing in $\Phi^{(0)}_h(\vec{k},1)$ multiplied by a factor of $\vec{k}$. Thus functional derivatives either bring down factors of $\vec{k}$ from the exponential or increase the order of cumulants which do not involve the one-point cumulant. To see how this works consider the set of derivatives

$$U_{\ldots\alpha_3\alpha_4}(1;34)=\frac{\delta^2}{\delta h_{\alpha_3}(3)\,\delta h_{\alpha_4}(4)}U^{(0)}_{\ldots}(1).$$

The functional derivatives then act on $\Phi^{(0)}_h(\vec{k},1)$. It is then easy enough to work out, using Eqs. (53) and (52), that

$$\frac{\delta^2}{\delta h_{\alpha_3}(3)\,\delta h_{\alpha_4}(4)}\Phi^{(0)}_h(\vec{k},1)|_{h=0}=\sum_{\alpha_1\alpha_2}k_{\alpha_1}k_{\alpha_2}[G^{(2)}_{\alpha_1\alpha_3}(13)G^{(2)}_{\alpha_2\alpha_4}(14)-\tfrac{1}{2}G^{(4)}_{\alpha_1\alpha_2\alpha_3\alpha_4}(1134)]\Phi^{(0)}_0(\vec{k},1).$$

We then have

$$U_{\ldots\alpha_3\alpha_4}(1;34)=\sum_{\alpha_1\alpha_2}U^{(0)}_{\ldots\alpha_1\alpha_2}(1)[G^{(2)}_{\alpha_1\alpha_3}(13)G^{(2)}_{\alpha_2\alpha_4}(14)-\tfrac{1}{2}G^{(4)}_{\alpha_1\alpha_2\alpha_3\alpha_4}(1134)]$$

and if $U^{(0)}_{\ldots}(1)$ is of $O(p)$ then $U^{(0)}_{\ldots\alpha_1\alpha_2}$ is of $O(p+1)$, $G^{(4)}_{\alpha_1\alpha_2\alpha_3\alpha_4}$ is of $O(1)$, and $U_{\ldots\alpha_3\alpha_4}(1;34)$ is of $O(p+1)$ plus higher-order terms.

Notice then that all derivatives of U and higher order terms can be expressed as products of the vertices $U^{(0)}$ with legs connected by cumulants. Any contribution to the nonlinear vertices Q_n and $\tilde{Q}_n$ can therefore be ordered summing up the contributions from each vertex and cumulant where a vertex with $2s+2$ legs makes a contribution of $O(s)$ and a cumulant $G^{(2p+2)}$ makes a contribution of $O(p)$.

For our purposes here, we only need two sets of the bare vertices $U^{(0)}$:

$$\mathcal{V}_s(\alpha_1;\alpha_2,\ldots,\alpha_{2s+2};1)=\int d^nx\ \hat{x}_{\alpha_1}\times\int\frac{d^nk}{(2\pi)^n}(i)k_{\alpha_2}k_{\alpha_3}\cdots k_{\alpha_{2s+2}}\Phi^{(0)}_h(\vec{k},1)e^{-i\vec{k}\cdot\vec{x}}$$

and

$$\tilde{\mathcal{V}}_s(\alpha_1,\alpha_2;\ldots,\alpha_{2s+2};1)=\int d^nx\ \hat{x}_{\alpha_1}\hat{x}_{\alpha_2}\times\int\frac{d^nk}{(2\pi)^n}k_{\alpha_3}k_{\alpha_4}\cdots k_{\alpha_{2s+2}}\Phi^{(0)}_h(\vec{k},1)e^{-i\vec{k}\cdot\vec{x}}.$$

It can be seen the $\mathcal{V}_s(\alpha_1;\alpha_2,\ldots,\alpha_{2s+2};1)$ is the natural generalization of the vertex $\phi_s(1)$ given by Eq. (92) in paper I. As in paper I, it is straightforward to work out these vertices explicitly by doing the integrations. The quantities we will need below are given by

$$\mathcal{V}_0(\alpha_1;\alpha_2;1)=\delta_{\alpha_1,\alpha_2}\sqrt{\frac{2}{S(1)}}\frac{\Gamma\left(\frac{n+2}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)},$$

$$\mathcal{V}_1(\alpha_1;\alpha_2,\alpha_3,\alpha_4;1)=I_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}\sqrt{\frac{2}{S^3(1)}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{n(n+2)\Gamma\left(\frac{n}{2}\right)},$$

$$\tilde{\mathcal{V}}_0(\alpha_1,\alpha_2;1)=\delta_{\alpha_1,\alpha_2}\frac{1}{n},$$

and

$$\tilde{\mathcal{V}}_1(\alpha_1,\alpha_2;\alpha_3,\alpha_4;1) = \frac{1}{S(1)}\left[\frac{1}{n}\delta_{\alpha_1,\alpha_2}\delta_{\alpha_1,\alpha_2} - \frac{1}{n+2}I_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}\right].$$

We have also introduced

$$S(1) \equiv G^{(2)}_{\alpha\alpha}(11) = \langle m^2_\alpha(1)\rangle,$$

with no summation over α, and the symmetric tensor

$$I_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} = \delta_{\alpha_1,\alpha_2}\delta_{\alpha_3,\alpha_4} + \delta_{\alpha_1,\alpha_3}\delta_{\alpha_2,\alpha_4} + \delta_{\alpha_1,\alpha_4}\delta_{\alpha_2,\alpha_3}.$$

A. Two-point nonlinear vertices to second order

Armed with these results we can return to the evaluation of the Q_n and $\tilde{Q}_n$ derived from Q_1 and $\tilde{Q}_1$ which are given by Eqs. (48) and (49). We summarize first the results for the two-point quantities which enter Eqs. (46) and (47). We find after a significant amount of work that

$$Q_2(12) = \Omega(1\bar{1})G(\bar{1}2) + \tilde{S}(1\overline{234})G(\overline{234}2), \quad (55)$$

$$\tilde{Q}_2(12) = \Omega(1\bar{1})G_{Mm}(\bar{1}2) + \tilde{S}_1(1\overline{234})G_{Mmmm}(\overline{234}2), \quad (56)$$

where integration and summation over repeated barred indices is implied. The various auxiliary quantities are defined by

$$\Omega(12) = [g_0(1) + g_1(1)S^{(2)}(1)]\mathcal{V}_0(\alpha_1;\alpha_2;1)\,\delta(12), \quad (57)$$

$$\tilde{S}(1234) = \frac{1}{3!}\tilde{V}_0(1234) + \tilde{V}_L(1234) + \tilde{V}_1(1234), \quad (58)$$

$$\tilde{S}_1(1234) = \frac{1}{2}\tilde{V}_0(1234) + \tilde{V}_L(1234) + \tilde{V}_2(1234) + \tilde{V}_3(1432) + \tilde{V}^T_L(1432), \quad (59)$$

$$\tilde{V}_0(1234) = -g_0(1)\mathcal{V}_1(\alpha_1;\alpha_2,\alpha_3,\alpha_4;1)\,\delta(12)\,\delta(13)\,\delta(14),$$

$$\tilde{V}_L(1234) = g_1(1)\mathcal{V}_0(\alpha_1;\alpha_2;1)\,\delta_{\alpha_3\alpha_4}W(134)\,\delta(12),$$

$$\tilde{V}_1(1234) = \tilde{\mathcal{V}}_1(\alpha_3,\alpha_4;\alpha_1,\alpha_2;1)\,\delta_{\alpha_3\alpha_4}W(134)\,\delta(12),$$

$$\tilde{V}_2(1234) = -W(134)\,\delta(12)\,\mathcal{Q}^{(0)}_{\alpha_3,\alpha_4,\alpha_1,\alpha_2}(1),$$

$$\tilde{V}_3(1234) = -2\tilde{\mathcal{V}}_1(\alpha_1,\alpha_3;\alpha_4,\alpha_2;1)\tilde{W}(134)\,\delta(12),$$

$$\tilde{V}^T_L(1234) = -2g_1(4)\mathcal{V}_0(\alpha_4;\alpha_2;4)\,\delta_{\alpha_1,\alpha_3}\tilde{W}(134)\,\delta(24),$$

and

$$\mathcal{Q}^{(0)}_{\alpha_3\alpha_4,\alpha_1,\alpha_2}(1) = -\tilde{\mathcal{V}}_1(\alpha_3,\alpha_4;\alpha_1,\alpha_2;1) + \tilde{\mathcal{V}}_1(\alpha_1,\alpha_3;\alpha_4,\alpha_2;1) + \tilde{\mathcal{V}}_1(\alpha_1,\alpha_4;\alpha_3,\alpha_2;1),$$

$$S^{(2)}(1) = \sum_i \langle \nabla_i m_\alpha(1)\nabla_i m_\alpha(1)\rangle,$$

where there is no summation over α. These results give a closed solution for the two-point correlation functions at zeroth order. This analysis of this lowest-order solution is given in the next section. In the following section we analyze the four-point correlation functions needed in order to extract the two-point correlation functions at second order.

VI. ZEROTH-ORDER THEORY FOR TWO-POINT CORRELATION FUNCTIONS

The equations of motion at zeroth order for the two-point correlation functions are given by the coupled set of Eqs. (46) and (47) with the lowest-order contributions for Q_2 and $\tilde{Q}_2$ given the leading-order terms in Eqs. (55) and (56). Inserting

$$Q^{(0)}_2(12) = \Omega(1\bar{1})G(\bar{1}2),$$

$$\tilde{Q}^{(0)}_2(12) = \Omega(1\bar{1})G_{Mm}(\bar{1}2)$$

into Eqs. (46) and (47) and explicitly writing the vector labels, we obtain

$$-i[\tilde{\Lambda}(1) + \omega_0(1)]G^{(0)}_{M_{\alpha_1}m_{\alpha_2}}(12) = \delta(12)\,\delta_{\alpha_1,\alpha_2},$$

$$i[\Lambda(1) - \omega_0(1)]G^{(0)}_{\alpha_1,\alpha_2}(12) = -\int d\bar{1}\;\Pi_0(1\bar{1})G^{(0)}_{M_{\alpha_1}m_{\alpha_2}}(\bar{1}2),$$

where we have defined

$$\omega_0(1) = [g_0(1) + g_1(1)S^{(2)}(1)]\mathcal{V}_0(\alpha_1,\alpha_1;1).$$

We see at once that the solution to this set of equations is diagonal in the vector indices,

$$G^{(0)}_{\alpha_1,\alpha_2}(12) = \delta_{\alpha_1,\alpha_2}G^{(0)}(12)$$

and

$$G^{(0)}_{M_{\alpha_1}m_{\alpha_2}}(12) = \delta_{\alpha_1,\alpha_2}G^{(0)}_{Mm}(12),$$

where $G^{(0)}(12)$ and $G^{(0)}_{Mm}(12)$ are the same quantities found in the scalar case in paper I. We summarize briefly the results since they are needed here. At long times we can write

$$\omega_0(t) = \frac{\omega}{t_c + t}, \quad (60)$$

where ω is a constant we will determine and t_c is a short-time cutoff which depends on details of the early-time evolution. One has then that the response function is given by

$$G_{Mm}^{(0)}(r,t_1t_2)=G_{Mm}^{(0)}(r,t_2t_1)$$
$$=-i\theta(t_2-t_1)\left(\frac{t_1+t_c}{t_2+t_c}\right)^{\omega}\frac{e^{-[r^2/4(t_2-t_1)]}}{[4\pi(t_2-t_1)]^{d/2}},$$

where $\vec{r}=\vec{r}_1-\vec{r}_2$, the correlation function is given by

$$G^{(0)}(r,t_1t_2)=g(0)\left(\frac{t_1+t_c}{t_c}\right)^{\omega}\left(\frac{t_2+t_c}{t_c}\right)^{\omega}\frac{e^{-r^2/8T}}{(8\pi T)^{d/2}}, \tag{61}$$

where $g(0)$ is the on-site value of the initial correlation function, and it is convenient to define

$$T=\frac{t_1+t_2}{2}.$$

If we are to have a self-consistent scaling equation then the autocorrelation function $(r=0)$, at large equal times $t_1=t_2=t$, must satisfy

$$S^{(0)}(t)=g(0)\left(\frac{t}{t_c}\right)^{2\omega}\frac{1}{(8\pi t)^{d/2}}\equiv A_0t.$$

Clearly this result fixes the exponent

$$\omega=\frac{1}{2}\left(1+\frac{d}{2}\right)$$

and the amplitude

$$A_0=\frac{1}{(t_c)^{2\omega}}\frac{g(0)}{(8\pi)^{d/2}}. \tag{62}$$

Equation (61) can be rewritten in the convenient form

$$G^{(0)}(r,t_1t_2)=\sqrt{S^{(0)}(t_1)S^{(0)}(t_2)}\Phi_{(0)}(t_1,t_2)e^{-1/2r^2/(4T)},$$

where

$$\Phi_{(0)}(t_1,t_2)=\left(\frac{\sqrt{t_1t_2}}{T}\right)^{d/2}.$$

The nonequilibrium exponent is defined in the long-time limit by

$$\frac{G^{(0)}(0,t_1,t_2)}{\sqrt{S^0(t_1)S^0(t_2)}}=\Phi_{(0)}(t_1,t_2)=\left(\frac{\sqrt{t_1t_2}}{T}\right)^{\lambda}$$

and we obtain the OJK result $\lambda=d/2$. Looking at equal times we have the auxiliary field scaling function

$$f_0(x)=\frac{G^{(0)}(r,tt)}{S^{(0)}(t)}=e^{-x^2/2},$$

where the scaled length is defined by $\vec{x}=\vec{r}/L(t)$, and the growth law is given by $L^2(t)=4t$. The exponent ν, defined by Eq. (71), is zero in the OJK approximation.

A. Four-point correlation functions at first order

If we are to evaluate $G^{(2)}$ and G_{Mm} to second order, we see that we must evaluate the four-point quantities $G^{(4)}$ and G_{Mmmm} to first order in the vertex expansion. This requires the evaluation of Q_4 and $\tilde{Q}_4$. Using the same techniques developed in evaluating Q_2 and $\tilde{Q}_2$ we find

$$\tilde{Q}_4(1234)=\Omega(1\bar{1})G_{Mmmm}(\bar{1}234)$$
$$+\tilde{S}_1(1\overline{234})P_M(\overline{234},234), \tag{63}$$

where $\Omega(12)$ and $\tilde{S}_1(1234)$ are defined by Eqs. (57) and (59), respectively, while

$$P_M(2'3'4',234)=G_{Mm}(4'2)[G(3'3)G(2'4)$$
$$+G(3'4)G(2'3)]+G_{Mm}(4'3)$$
$$\times[G(3'2)G(2'4)+G(3'4)G(2'2)]$$
$$+G_{Mm}(4'2)[G(3'2)G(2'3)$$
$$+G(3'3)G(2'2)]. \tag{64}$$

We also have

$$Q_4(1234)=\Omega(1\bar{1})G^{(4)}(\bar{1}234)+\tilde{S}(1\overline{234})P(\overline{234},234), \tag{65}$$

where $\tilde{S}(1234)$ is given by Eq. (58) and

$$P(2'3'4',234)=G(4'2)[G(3'3)G(2'4)$$
$$+G(3'4)G(2'3)]+G(4'3)$$
$$\times[G(3'2)G(2'4)+G(3'4)G(2'2)]$$
$$+G(4'2)[G(3'2)G(2'3)$$
$$+G(3'3)G(2'2)].$$

Inserting Eq. (63) for $\tilde{Q}_4$ into Eq. (42) we see that we can do a partial integration and write

$$G_{Mmmm}(1234)=G_{Mm}^{(0)}(1\bar{1})i\tilde{S}_1(\overline{1234})P_M(\overline{234},234), \tag{66}$$

where $\tilde{S}_1$ is defined by Eq. (66). Using the symmetry properties of $P_M(\overline{234},234)$ we can show that Eq. (66) can be written as

$$G_{Mmmm}(1234)=G_{Mm}^{(0)}(1\bar{1})V_s(\bar{2};\overline{134})P_M(\overline{234},234), \tag{67}$$

where

$$V_s(2;134)=\frac{i}{2}\tilde{V}_0(1234)+i\tilde{V}_{L,s}(2,134)+i\tilde{V}_{1,s}(2,134)$$

and the symmetrized vertices are given by

$$\tilde{V}_{L,s}(2,134)=\tilde{V}_L(2134)+\tilde{V}_L(2,413)+\tilde{V}_L(2,314)$$

and

$$\tilde{V}_{1,s}(2,134)=\tilde{V}_1(2134)+\tilde{V}_1(2,413)+\tilde{V}_1(2,314).$$

In turn Eqs. (67) and (65) can be put back into Eq. (43) with $n=4$ to obtain $G^{(4)}$. After manipulations it can be written in the properly symmetric form

$$G^{(4)}(1234)=\frac{1}{3}G^{(0)}_{mM}(1\overline{1})V_s(\overline{1};\overline{234})P(\overline{234},234)+\frac{1}{3}G^{(0)}_{mM}(2\overline{1})V_s(\overline{1};\overline{234})P(\overline{234},134)+\frac{1}{3}G^{(0)}_{mM}(3\overline{1})V_s(\overline{1};\overline{234})P(\overline{234},124)+\frac{1}{3}G^{(0)}_{mM}(4\overline{1})V_s(\overline{1};\overline{234})P(\overline{234},123).$$

We see after these manipulations that all of the first-order corrections have been combined into a single vertex.

VII. TWO-POINT CORRELATION FUNCTION AT SECOND ORDER

A. General equations

Given $G^{(4)}$ and G_{Mmmm} at first order, we can return to Eqs. (63) and (56) to obtain Q_2 and $\tilde{Q}_2$ to second order. These in turn are put back into Eqs. (46) and (47) to obtain the second-order results for the two-point correlation functions. We focus here on the correlation function. After a single integration of Eq. (47) we have

$$G(12)=G^{(0)}(12)+G^{(0)}_{mM}(1\overline{1})\frac{1}{3}V_s(\overline{1};\overline{234})G_4(\overline{2342})+G^{(0)}(1\overline{1})V_s(\overline{2};\overline{134})G_{Mmmm}(\overline{2342})$$
$$=G^{(0)}(12)+G^{(0)}_{mM}(1\overline{1})\frac{1}{3}V_s(\overline{1};\overline{234})\frac{1}{3}V_s(\overline{1'};\overline{2'}\overline{3'}\overline{4'})[G^{(0)}_{mM}(2\overline{1'})P(\overline{2'}\overline{3'}\overline{4'},\overline{234})+G^{(0)}_{mM}(\overline{21'})P(\overline{2'}\overline{3'}\overline{4'},2\overline{34})$$
$$+G^{(0)}_{mM}(\overline{31'})P(\overline{2'}\overline{3'}\overline{4'},\overline{22}4)+G^{(0)}_{mM}(\overline{41'})P(\overline{2'}\overline{3'}\overline{4'},\overline{22}3)]$$
$$+G^{(0)}(1\overline{1})V_s(\overline{2};\overline{134})G^{(0)}_{Mm}(\overline{21'})V_s(\overline{2'};\overline{1'}\overline{3'}\overline{4'})P_M(\overline{2'}\overline{3'}\overline{4'},2\overline{34}).$$

This last term simplifies since, because of causality, only the term proportional to $G_{Mm}(\overline{2'}2)$ in $P_M(\overline{2'}\overline{3'}\overline{4'},2\overline{34})$ survives, and

$$V_s(\overline{1'};\overline{2'}\overline{3'}\overline{4'})P(\overline{2'}\overline{3'}\overline{4'},\overline{234})=6V_s(\overline{1'};\overline{2'}\overline{3'}\overline{4'})G(\overline{2'}\overline{2})G(\overline{3'}\overline{3})G(\overline{4'}\overline{4}).$$

We then have the final formal expressions

$$G(12)=G^{(0)}(12)+G^{(S)}(12)+G^{(U)}(12)+G^{(U)}(21),$$

where the *symmetric* contributions are given by

$$G^{(S)}(12)=\frac{2}{3}G^{(0)}_{mM}(1\overline{1})G^{(0)}_{mM}(2\overline{1'})V_s(\overline{1};\overline{234})\times V_s(\overline{1'};\overline{2'}\overline{3'}\overline{4'})G^{(0)}(\overline{2'}\overline{2})\times G^{(0)}(\overline{3'}\overline{3})G^{(0)}(\overline{4'}\overline{4})$$

and the *unsymmetric* contributions are given by

$$G^{(U)}(12)=2G^{(0)}_{mM}(1\overline{1})G^{(0)}(2\overline{2'})V_s(\overline{1};\overline{234})\times V_s(\overline{1'};\overline{2'}\overline{3'}\overline{4'})G^{(0)}_{mM}(\overline{21'})\times G^{(0)}(\overline{3'}\overline{3})G^{(0)}(\overline{4'}\overline{4}).$$

The detailed analysis of these contributions to the correlation function follows closely the analysis developed in detail in paper I. Indeed if the full vertex V_s is replaced by $(i/2)\tilde{V}_0$ and n set to 1, these equations reduce to those found in paper I. One can again carry out explicitly the internal spatial integrations. Among the new elements in this analysis is the treatment of the gradient insertions in the vertices and the internal vector sums. One must also introduce the parameter

$$g=g_1(1)\mathcal{V}_0(\alpha_1;\alpha_1:1)S(1),$$

which, as anticipated earlier, requires that $g_1(1)$ go as L^{-1} for long times with an amplitude which is determined as part of the scaling structure.

B. Extraction of indices

As in paper I, all of the various logarithmic singularities found in second order, and arising from internal time integrations, can be absorbed into expressions for the exponents ω, λ, and ν. At second order in the vertex expansion the exponents are determined by the set of equations

$$\lambda=\frac{d}{2}+\omega^2\frac{2^{d+1}}{3^{d/2}}H_S \tag{68}$$

and

$$\frac{\nu}{2}=\omega^2 2^{d+1}\left[H_U+\frac{H_S}{3^{d/2}}\right], \tag{69}$$

where the quantities H_U and H_S are given below. The condition that the growth law $L\approx t^{1/2}$ be maintained order by order in perturbation theory determines the parameter ω, and, as in paper I, can be expressed in terms of the exponent ν:

$$2\omega+\frac{\nu}{2}=1+d/2. \tag{70}$$

There are two intrinsically different contributions, H_U and H_S, to the second-order expressions for the correlation functions which come from graphs with different structures. H_U and H_S, which depend only on the parameters ω, g, d, and n, are defined in terms of a set of auxiliary quantities:

$$H_S=Q_S^{(0)}M_d+Q_S^{(2)}M_{d+2},$$

$$H_U=Q_U^{(0)}K_d+Q_U^{(6)}K_d^{(6)}+Q_U^{(9)}K_d^{(9)},$$

where

$$Q_S^{(0)}=\frac{1}{2(n+2)}\left(1-\frac{d}{\omega}S^{(2)}\right),$$

$$Q_S^{(2)}=\frac{d}{18\omega^2}[(d+5)S^{(3)}+2(d-1)S^{(4)}],$$

$$Q_U^{(0)}=\frac{3}{2(n+2)}\left(1-\frac{d}{\omega}S^{(2)}\right),$$

$$Q_U^{(6)}=\frac{2}{\omega^2}S^{(3)},$$

$$Q_U^{(9)}=\frac{4}{\omega^2}S^{(4)},$$

and the $S^{(i)}$ are basically the result of internal n-vector sums,

$$S^{(2)}=-\frac{2(n-1)}{3n}+\frac{(n+2)}{3}g,$$

$$S^{(3)}=\frac{2(n-1)}{n(n+2)}+ng^2,$$

$$S^{(4)}=\frac{(n-2)(n-1)}{n^2(n+2)}-\frac{2(n-1)}{n}g+g^2.$$

The constants g and ω parametrize the scaling properties of the nonlinear terms in the equation of motion for the auxiliary field. Finally, we have the d-dependent integrals

$$M_d=\int_0^1 dz\,\frac{z^{d/2-1}}{[1+z]^d}=\frac{1}{2}\,\frac{\Gamma^2(d/2)}{\Gamma(d)}, \tag{71}$$

$$K_d^{(0)}=\int_0^1 dz\,\frac{z^{d/2-1}}{[(1+z)(3-z)]^{d/2}}, \tag{72}$$

$$K_d^{(6)}=\frac{d}{4}\int_0^1 dz\frac{z^{d/2-1}}{[(1+z)(3-z)]^{d/2}}\,\frac{z}{(1+z)^2}\times\left[1-\frac{2(1-z)}{(3-z)}+(d+2)\frac{(1-z)^2}{(3-z)^2}\right],$$

$$K_d^{(9)}=-\frac{d}{4}\int_0^1 dz\frac{z^{d/2-1}}{[(1+z)(3-z)]^{d/2}}\,\frac{z}{(1+z)(3-z)}\times\left[1-(d+2)\frac{(1-z)}{(3-z)}\right]. \tag{73}$$

If we set $n=1$ and $g=0$ then this set of equations reduces to that found in I. Then Eqs. (69) and (70) can be solved for ω and ν and the results inserted in Eq. (68) to obtain the index λ as given in paper I.

The parameters ω and g should be thought of as being generated by some type of renormalization group (RG) analysis. Carrying forward the RG analogy, these parameters are to be determined as part of finding the scaling fixed point in the problem. This process is similar to finding a fixed point Hamiltonian in critical phenomena. The parameter ω occurs naturally at lowest order in the perturbation theory expansion, while g naturally arises at second order in the expansion. ω and g are determined by the requirements that the scaling law $L\approx t^{1/2}$ and the index relation $\nu=2\lambda-d$ hold at all orders. The maintenance of the growth law leads to the condition given by Eq. (70). The requirement $\nu=2\lambda-d$ is enforced by choosing

$$H_U=0. \tag{74}$$

While there are many possible ways of extracting explicit numbers for the indices from the perturbation expansion just described, we discuss two here. In the *expansion* method we set $\omega=\omega_0=\frac{1}{2}(1+d/2)$ in the second-order terms and obtain the indices directly. In the second method we look for a self-consistent solution of Eqs. (70) and (74) for g and ω. For large d and n these two approaches are equivalent. While the various d-dependent integrals, $K_d^{(i)}$, etc. can be worked out analytically for specific values of d, the expressions are not very illuminating. Numerical values for λ, ν, and g are given in Tables I, II, and III. Except for the values of λ maked by an *, whose significance is discussed below, the values of ν are given by $\nu=2\lambda-d$ and ω is given by Eq. (74). One sees that the self-consistent values for λ are all close to the OJK values. The perturbation theory results can lead to much larger corrections.

It is instructive to work out the large-d limit analytically. For general n, one finds a solution of Eq. (74) in the limit with

$$g=\frac{3}{(n+2)}\left(\frac{1}{4}+\frac{2(n-1)}{3n}\right).$$

This expression for g has a minimum value of 1/4 for $n=1$, a maximum of 7/16 for $n=2$, and then a slow decay to

TABLE I. Values of exponent λ. In the second column per refers to values from the current theory fully expanded, sel refers to a self-consistent solutions from the current theory, TUG refers to values from Ref. [4], OJK refers to the values from Ref. [3], and num refers to numerically determined values. An asterisk indicates that no solution to Eq. (74) was found.

Dimension		$n=1$	$n=2$	$n=3$
	per	0.5819	0.8120	1.1172*
	sel	0.5154	0.5590	0.6206*
1	TUG	1.0	0.699	0.622
	OJK	0.5	0.5	0.5
	num			0.648[b]
	per	1.0530	1.2045	1.5326*
	sel	1.0059	1.0227	1.0597*
2	TUG	1.2887	1.171	1.117
	OJK	1.0	1.0	1.0
	num	1.246 ± 0.02[a]		
	per	1.5375	1.6284	1.8240
	sel	1.5024	1.5082	1.5216
3	TUG	1.6726	1.618	1.587
	OJK	1.5	1.5	1.5
	num	1.838 ± 0.2[a]		
large		$d/2$	$d/2$	$d/2$

[a]From Ref. [10].
[b]From Ref. [15].

zero as a function of n. For the scalar case the contributions to ν and $\lambda=d/2+\nu/2$ are given to leading order by

$$\nu=\frac{\sqrt{2\pi}}{96}\frac{d^{3/2}}{3^{d/2}},$$

which gives exponential decay to zero for large d. For large n the exponents are also given by the OJK result, with corrections of the form $\lambda=d/2+\lambda_1(d)/n+\cdots$, where the precise dependence of λ_1 on d is complicated.

This procedure for fixing the coefficients ω and g works straightforwardly for the scalar case and generally for $d>n$. However, for $d<n$ one finds, for small enough d, that the new spin-wave contributions [proportional to $(n-1)$ in the $S^{(i)}$] lead to a breakdown in this process. Solutions to Eq. (74) do not exist and one cannot enforce the relation $\nu=2\lambda-d$. In this case we have chosen g such that H_U is a minimum. The structure of the theory for $n>d$ needs further work. This is just the regime where one does not generate stable topological defects.

VIII. CONCLUSIONS

It has been shown how one can extend the method developed previously for a scalar order parameter to the case of the n-vector model. The approach developed here appears to be a rather general tool for looking at field theories where the field is growing and showing scaling behavior. One is able to develop a systematic expansion in the number of labels on the nonlinear vertices appearing in the problem. This expansion leads directly to expressions for the anomalous dimensions in the problem. Less generally one is then confronted with the interpretation of the perturbation theory expansion in a particular realization of the theory. As organized here, the self-consistent corrections to the OJK results for the indices λ and ν are typically quite small and vanish for both large d and n. The large d convergence is tied to the enforcement of the equation $\nu=2\lambda-d$ relating the indices.

The transverse degrees of freedom enter quite differently into the problem compared with the longitudinal degrees of freedom. The longitudinal contributions to the nonlinear terms in the equation of motion for the auxiliary field $\vec{m}$ must be determined self-consistently in constructing the scaling properties in the problem. The contributions to the transverse part of the equation of motion for the auxiliary field are, because the transverse degrees of freedom are massless, fixed by the requirement, for self-consistency, that the amplitude for the transverse order-parameter fluctuations be small compared to the ordered component. It turns out that the transverse contributions to the equation of motion for the auxiliary field are sufficiently strong, for fixed $n>3$ and sufficiently small d, that we are unable to enforce the condition $\nu=2\lambda-d$. This regime requires further study.

The point of view developed here is somewhat unanticipated. In the most direct approach, as discussed in some detail in paper I, one makes the substitution $\vec{\sigma}=\vec{\sigma}[\vec{m}]$ into the order-parameter equation of motion to obtain the equation of motion for $\vec{m}$. The path taken here is quite different since the equation of motion for $\vec{m}$ is constructed self-consistently. The surprising point is that the quantity $\Xi_L(\vec{m})$, which enters the equation of motion satisfied by the auxiliary field, is *not* determined when we insert $\vec{\psi}=\vec{\sigma}+\vec{u}$ into the order-parameter equation of motion. It is this freedom that allows us to construct the scaling regime form for $\Xi_L(\vec{m})$.

TABLE III. Values of exponent ν. An asterisk indicates that no solution to Eq. (74) was found.

Dimension	$n=1$	$n=2$	$n=3$
1	3.08732×10^{-2}	1.181836×10^{-1}	1.7438*
2	1.17040×10^{-2}	4.530440×10^{-2}	1.3928*
3	4.80000×10^{-3}	1.645200×10^{-2}	4.3149×10^{-2}

TABLE II. Values of parameter g. An asterisk indicates that no solution to Eq. (74) was found.

Dimension	$n=1$	$n=2$	$n=3$
1	$7.6423632242\times10^{-1}$	1.2448881672	1.3940000000*
2	$5.1175087572\times10^{-1}$	$9.3442756917\times10^{-1}$	1.2310000000*
3	$4.2674144077\times10^{-1}$	$7.9300891113\times10^{-1}$	1.0569791347
large	0.25	0.4375	0.416666

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